SYDNEY GRAMMAR SCHOOL

2014 Half-Yearly Examination

FORM VI
MATHEMATICS EXTENSION 2

Wednesday 26th February 2014

General Instructions
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks
- All questions may be attempted.

Section I – 10 Marks
- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 60 Marks
- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection
- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist
- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 81 boys

Examiner
RCF

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The solutions to the quadratic equation $(z-2i)(z-1-i)=0$ are: **1**

(A) $z = 2i$ or $-1-i$ (B) $z = 1+i$ or $2i$

(C) $z = -2i$ or $-1-i$ (D) $z = 1+i$ or $-2i$

QUESTION TWO

A primitive of $e^{2x}\sin(e^{2x})$ is: **1**

(A) $2\cos(e^{2x})$ (B) $\frac{1}{2}\sin(e^{2x})$

(C) $-2\sin(e^{2x})$ (D) $-\frac{1}{2}\cos(e^{2x})$

QUESTION THREE

Which of the following represents the complex number $2\sqrt{3}-2i$ expressed in modulus–argument form? **1**

(A) $4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ (B) $4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(C) $4\left(\cos(-\frac{\pi}{6})+i\sin(-\frac{\pi}{6})\right)$ (D) $4\left(\cos(-\frac{\pi}{3})+i\sin(-\frac{\pi}{3})\right)$

QUESTION FOUR

Which of the following is a correct primitive of $\dfrac{1}{\sqrt{16-9x^2}}$? **1**

(A) $\frac{1}{3}\sin^{-1}\left(\dfrac{3x}{4}\right)$ (B) $\frac{1}{4}\sin^{-1}(3x)$

(C) $4\sin^{-1}\left(\dfrac{3x}{4}\right)$ (D) $\frac{4}{3}\sin^{-1}\left(\dfrac{4x}{3}\right)$

Exam continues next page ...

QUESTION FIVE

Given $x^3 - 2xy^2 = 10$, the correct result for $\dfrac{dy}{dx}$ is: **1**

(A) $\dfrac{3x^2 - 2y^2}{4xy}$

(B) $\dfrac{3x}{4}$

(C) $\dfrac{3x}{4y}$

(D) $\dfrac{2y^2 - 3x^2}{4xy}$

QUESTION SIX

Which of the following is the correct result for $\displaystyle\int \frac{1}{x^2 - 4x + 8}\, dx$? **1**

(A) $2\tan^{-1}(x-2) + c$

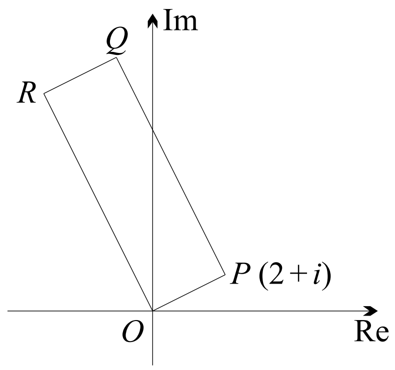
(B) $\tan^{-1}\left(\dfrac{x-2}{2}\right) + c$

(C) $\dfrac{1}{2}\tan^{-1}(x-2) + c$

(D) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x-2}{2}\right) + c$

QUESTION SEVEN

1



The diagram above shows a rectangle $OPQR$ in the complex plane. The vertex P represents the complex number $2 + i$ and the side OR is three times the length of OP. The complex number that corresponds to vertex R is:

(A) $-6 - 3i$

(B) $3 + 6i$

(C) $6i - 3$

(D) $3i - 6$

Exam continues overleaf ...

QUESTION EIGHT

Which of the following statements about the inverse trigonometric functions is false? **1**

(A) $\dfrac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0$

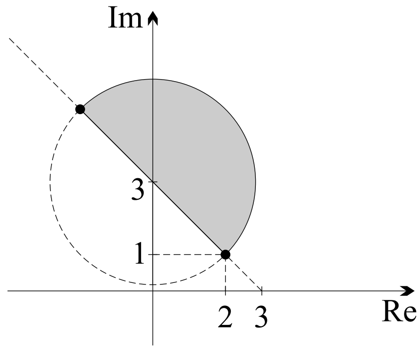
(B) The range of $y = \cos^{-1} 3x$ is $0 \le y \le \pi$.

(C) The inverse sine graph has a stationary point of inflection at the origin.

(D) $f(x) = \tan^{-1}(-x)$ is a purely decreasing function.

QUESTION NINE

1



The shaded region in the diagram above illustrates the intersection of which of the following pairs of complex inequalities?

(A) $|z - 3| \le 8$ and $0 \le \arg(z - 3i) \le \dfrac{3\pi}{4}$

(B) $|z - 3i| \le 8$ and $0 \le \arg(z - 3) \le \dfrac{3\pi}{4}$

(C) $|z - 3| \le 2\sqrt{2}$ and $0 \le \arg(z - 3i) \le \dfrac{3\pi}{4}$

(D) $|z - 3i| \le 2\sqrt{2}$ and $0 \le \arg(z - 3) \le \dfrac{3\pi}{4}$

Exam continues next page ...

QUESTION TEN

A given function, $f(x)$, has period 2, that is $f(x) = f(x+2)$. **1**

Which of the following is certain to equal $\displaystyle\int_{-1}^{1} f(x)\,dx$?

(A) $\displaystyle\int_{0}^{2} f(x)\,dx$

(B) 0

(C) $\displaystyle 2\int_{0}^{1} f(x)\,dx$

(D) $\displaystyle\int_{1}^{3} f(x)\,dx$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = 3 - 4i$ and $w = 5 + 2i$.

(i) Find $2z - 3w$. **1**

(ii) Find $\overline{wz}$. **1**

(iii) Express $\dfrac{w}{z}$ in the form $a + ib$, where a and b are real numbers. **2**

(b) Given that $z = 1 + 2i$ is a solution of the quadratic equation $z^2 - kz + (6 + 2i) = 0$, find the value of k. **2**

(c) (i) Find the two square roots of $8 + 6i$. **1**

(ii) Hence, or otherwise, solve the quadratic equation $iz^2 + (1 + i)z + (-1 + 2i) = 0$. **2**

(d) A complex number z satisfies $|z - 2 + 5i| = 1$.

(i) Sketch the locus of z in the Argand diagram. **1**

(ii) Hence find the minimum value of $|z|$. **1**

(e) Let $w_1 = 1 + i\sqrt{3}$ and $w_2 = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

(i) Express w_1 in modulus−argument form. **1**

(ii) By considering the product $w_1 w_2$, find the exact value of $\sin\frac{7\pi}{12}$. **3**

Exam continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Find $\displaystyle\int_0^3 \frac{x-1}{\sqrt{x+1}}\,dx$ by using the substitution $u = \sqrt{x+1}$. **3**

(b) (i) Find the values of A, B and C such that **2**

$$\frac{5x^2+7x+8}{(2x-1)(x^2+4)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+4}\,.$$

(ii) Hence find $\displaystyle\int \frac{5x^2+7x+8}{(2x-1)(x^2+4)}\,dx$. **2**

(c) Use the t-formulae to find $\displaystyle\int \frac{dx}{\sin x(1+\cos x)}$. **2**

(d) Use integration by parts to find $\displaystyle\int e^{-x}\cos x\,dx$. **3**

(e) (i) Show that $\displaystyle\frac{d}{dx}\ln\left(x+\sqrt{x^2+4}\right) = \frac{1}{\sqrt{x^2+4}}$. **1**

(ii) Hence, or otherwise, show that **2**

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}\,dx = 2\ln\left(\frac{\sqrt{5}+1}{2}\right).$$

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) A locus in the Argand diagram is described by $z\overline{z} = \Big(\text{Re}(z+2)\Big)^2$. Determine the Cartesian equation of the locus. **2**

(b) (i) Differentiate $f(x) = \tan^{-1} x + \tan^{-1} \dfrac{1}{x}$, where $x > 0$. **1**

(ii) Hence find the exact value of $\tan^{-1} 2 + \tan^{-1} \dfrac{1}{2}$. **1**

(c) Use mathematical induction to prove that for all positive integer values of n: **3**

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

(d) Use the product to sum result $\sin A \sin B = -\frac{1}{2}\Big(\cos(A+B) - \cos(A-B)\Big)$ **2**

to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin 7x \sin 3x \, dx$.

(e) Find the gradient of the tangent to the curve $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ at the point $\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right)$. **2**

(f) Let $u_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^{2n}\theta \, d\theta$, where $n \geq 0$.

(i) Show that $u_n = \dfrac{1}{2n-1} - u_{n-1}$, where $n \geq 1$. **2**

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^6\theta \, d\theta$. **2**

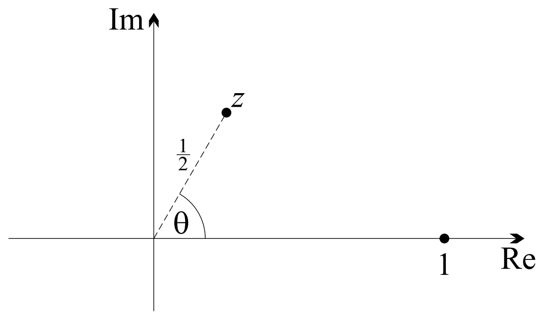
QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Use the results for $\sin(A \pm B)$ to prove that **1**

$$\sin C + \sin D = 2\sin\tfrac{1}{2}(C+D)\cos\tfrac{1}{2}(C-D).$$

(ii) Hence, or otherwise, find the general solution of $\sin 3x + \sin x = \cos x$. **3**

(b) Consider the sequence $1,\ z,\ z^2,\ z^3,\ \ldots,$ where $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ and θ is acute.



(i) Copy the Argand diagram given above into your answer booklet and add points to represent z^2 and z^3. **1**

(ii) Explain why $\text{Im}(z^n) = (\frac{1}{2})^n \sin n\theta$. **1**

(iii) The series $1 + z + z^2 + z^3 + \cdots$ converges to a limit. **3**

Show that $\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \cdots = \dfrac{2\sin\theta}{5 - 4\cos\theta}$.

(c) (i) Show that $\dfrac{1}{x^4+4} = \dfrac{x+2}{8(x^2+2x+2)} - \dfrac{x-2}{8(x^2-2x+2)}$. **1**

(ii) Consider the integral $I = \displaystyle\int_0^N \frac{1}{x^4+4}\,dx$. **3**

Show that $I = \dfrac{1}{16}\left(\log\left(\dfrac{N^2+2N+2}{N^2-2N+2}\right) + 2\tan^{-1}(N+1) + 2\tan^{-1}(N-1)\right)$.

(iii) Hence evaluate $\displaystyle\lim_{N\to\infty}\int_{-N}^{N} \frac{1}{x^4+4}\,dx$. **2**

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} \, x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \;\; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} \, e^{ax}, \;\; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$\text{NOTE}: \;\; \ln x = \log_e x, \;\; x > 0$$

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Candidate number: ....

Question One
A ◯ B ◯ C ◯ D ◯

Question Two
A ◯ B ◯ C ◯ D ◯

Question Three
A ◯ B ◯ C ◯ D ◯

Question Four
A ◯ B ◯ C ◯ D ◯

Question Five
A ◯ B ◯ C ◯ D ◯

Question Six
A ◯ B ◯ C ◯ D ◯

Question Seven
A ◯ B ◯ C ◯ D ◯

Question Eight
A ◯ B ◯ C ◯ D ◯

Question Nine
A ◯ B ◯ C ◯ D ◯

Question Ten
A ◯ B ◯ C ◯ D ◯

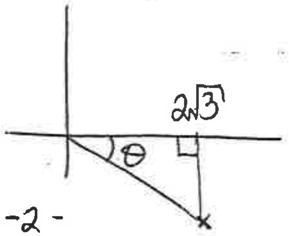
4U Half Yearly 2014

Multiple Choice

① $z = 2i$ or $1+i$ (B) ✓

② $\int e^{2x}\sin(e^{2x})\,dx = -\frac{1}{2}\cos e^{2x}$ (D) ✓

③



$r^2 = (-2)^2 + (2\sqrt{3})^2$
$= 4 + 12$
$= 16.$
$r = 4$
$\theta = -\tan^{-1}\left(\frac{2}{2\sqrt{3}}\right)$
$= -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
$= -\frac{\pi}{6}.$

ie $4\operatorname{cis}\left(-\frac{\pi}{6}\right)$ (C) ✓

④ $\int \frac{1}{\sqrt{16-9x^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{4}{3}\right)^2 - x^2}}\,dx$
$= \frac{1}{3}\sin^{-1}\left(\frac{x}{4/3}\right) + C$
$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right) + C$ (A) ✓

⑤ $x^3 - 2xy^2 = 10$
Differentiate implicitly
$3x^2 - 2y^2 - 2x\left(2y\frac{dy}{dx}\right) = 0$
$3x^2 - 2y^2 = 4xy\frac{dy}{dx}$
$\frac{dy}{dx} = \frac{3x^2 - 2y^2}{4xy}$ (A) ✓

⑥ $\int \frac{1}{x^2-4x+8}\,dx = \int \frac{1}{(x-2)^2+4}\,dx$
$= \frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + C$ (D) ✓

⑦ P represents $(2+i)$
rotation by 90° anticlockwise ⇒ multiplication by i
enlargement with factor of 3 radially ⇒ $\times 3$
$\therefore 3i(2+i) = 6i - 3$ (C) ✓

⑧ $\frac{d}{dx}(\sin^{-1}x + \cos^{-1}x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$ TRUE

$y = \cos^{-1}3x$ Stretch in x direction with factor $\frac{1}{3}$
Range $0 \leq y \leq \pi$ TRUE

$y = \sin^{-1}x$ Has point of inflection at origin but gradient is one ie non stationary FALSE

$f(x) = \tan^{-1}(-x)$ Inverse tan reflected in y-axis $\therefore$ decreasing TRUE

(C) ✓

⑨ $|z - 3i| \leq 8$ radius $= \sqrt{2^2+2^2}$
$= \sqrt{8}$
and $0 \leq \arg(z-3) \leq \frac{3\pi}{4}$ (D) ✓

⑩ $\int_{-1}^{1} f(x)\,dx = \int_{1}^{3} f(x)\,dx$ since periodicity 2 (D) ✓

Total 10/10

11 a) $z = 3-4i \quad w = 5+2i$

(i) $2z - 3w = 2(3-4i) - 3(5+2i)$
$= 6-8i-15-6i$
$= -9-14i$ ✓

(ii) $\overline{wz} = \overline{(3-4i)(5+2i)}$
$= \overline{15-20i+6i-8i^2}$
$= \overline{23-14i}$
$= 23+14i$ ✓

(iii) $\frac{w}{z} = \frac{5+2i}{3-4i} \; \frac{(\times 3+4i)}{(\times 3+4i)}$ ✓
$= \frac{15+20i+6i+8i^2}{25}$
$= \frac{7+26i}{25}$ ✓ $= \frac{7}{25} + \frac{26}{25}i$ (ie $a = \frac{7}{25}$ $b = \frac{26}{25}$)

(b) $1+2i$ a solution $z^2 - kz + (6+2i) = 0$
∴ satisfies LHS = 0
$(1+2i)^2 - k(1+2i) + (6+2i) = 0$ ✓
$-3+4i-k-2ki+6+2i = 0$
$(3-k) + (6-2k)i = 0$
∴ $k = 3$. ✓

(c) (i) Let $(a+bi)^2 = 8+6i$
∴ $a^2 - b^2 = 8$ ①
$2ab = 6 \Rightarrow ab = 3$ ②
by inspection $a=3$ $b=1$
or $a=(-3)$ $b=(-1)$.
∴ square roots are $3+i$ or $-(3+i)$ ✓

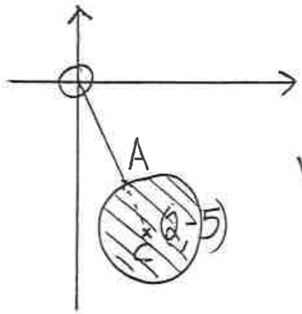
(ii) $iz^2 + (1+i)z + (-1+2i) = 0$

$\Delta = (1+i)^2 - 4 \times i \times (-1+2i)$
$= 2i + 4i - 8i^2$
$= 8+6i$ ✓

∴ $z = \frac{-(1+i)+3+i}{2i}$ or $\frac{-(1+i)-(3+i)}{2i}$
$= \frac{2}{2i}$ $\qquad = \frac{-4-2i}{2i}$
$= \frac{1}{i}$ $\qquad = \frac{-2-i}{i} \; \frac{(\times -i)}{(\times -i)}$
$= -i$ $\qquad = -1+2i$ ✓

d)

$|z-2+5i| \le 1$ ○ Centre $(2,-5)$ Radius 1
Inside Region

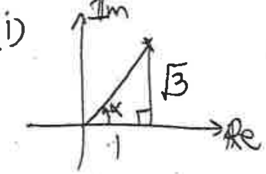


✓

$OC = \sqrt{2^2+5^2}$
$= \sqrt{29}$
∴ $\min|z| = OA$
$= \sqrt{29} - 1$ u ✓

e) $\omega_1 = 1 + i\sqrt{3}$ $\quad \omega_2 = 2\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

(i)



$\omega_1 = r\,\text{cis}\,\alpha$

$r^2 = 1^2 + (\sqrt{3})^2$

$r = 2$

$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

$\therefore \omega_1 = 2\,\text{cis}\,\frac{\pi}{3}$ ✓

(ii) $\omega_1\omega_2 = 2\,\text{cis}\,\frac{\pi}{3} \times 2\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

$= 4\sqrt{2}\,\text{cis}\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$= 4\sqrt{2}\,\text{cis}\left(\frac{7\pi}{12}\right)$ ✓

$\omega_1\omega_2 = (1 + i\sqrt{3}) \times (2 + 2i)$

$= 2 + 2\sqrt{3}\,i + 2i - 2\sqrt{3}$

$= 2 - 2\sqrt{3} + (2 + 2\sqrt{3})i$ ✓

$\therefore 4\sqrt{2}\sin\frac{7\pi}{12} = 2 + 2\sqrt{3}$

$\therefore \sin\frac{7\pi}{12} = \frac{2 + 2\sqrt{3}}{4\sqrt{2}}$ ✓

$= \frac{1 + \sqrt{3}}{2\sqrt{2}}$

$= \frac{\sqrt{2} + \sqrt{6}}{4}$

(15)

12 a) $\int_0^3 \frac{x-1}{\sqrt{x+1}}$

$u = \sqrt{x+1}$

$du = \frac{1}{2}(x+1)^{-\frac{1}{2}}dx$

$2du = \frac{dx}{\sqrt{x+1}}$ ✓

$u^2 - 1 = x$

x	0	3
u	1	2

$= \int_1^2 [(u^2-1) - 1] \times 2du$ ✓

$= \int_1^2 2u^2 - 4\,du$ ✓

$= \left[\frac{2u^3}{3} - 4u\right]_1^2$

$= \left(\frac{2}{3} \times 8 - 8\right) - \left(\frac{2}{3} - 4\right)$

$= \frac{14}{3} - 4$

$= \frac{2}{3}$ ✓

b) $\frac{5x^2 + 7x + 8}{(2x-1)(x^2+4)} = \frac{A}{(2x-1)} + \frac{Bx + C}{x^2 + 4}$

$= \frac{Ax^2 + 4A + (Bx + C)(2x - 1)}{(2x-1)(x^2+4)}$

$= \frac{(A + 2B)x^2 + (2C - B)x + (4A - C)}{(2x-1)(x^2+4)}$

Comparing coeffs

x^2; $A + 2B = 5$ ①

x^1; $2C - B = 7$ ②

x^0; $4A - C = 8$ ③ ✓

①+2×② $A + 4C = 19$ ④

4×③+④ $17A = 51$

$A = 3$

$\therefore B = 1$

$\therefore C = 4$ ✓

$\therefore LHS = \frac{3}{2x-1} + \frac{x+4}{x^2+4}$

(ii) $\int \frac{5x^2 + 7x + 8}{(2x-1)(x^2+4)} = \int \frac{3}{2x-1} + \frac{x}{x^2+4} + \frac{4}{x^2+4}\,dx$ ✓ One Correct Primitive

$= \frac{3}{2}\ln|2x-1| + \frac{1}{2}\ln(x^2+4) + 4 \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

$= \frac{1}{2}\left(3\ln|2x-1| + \ln(x^2+4) + 4\tan^{-1}\left(\frac{x}{2}\right)\right) + C$ ✓ Three Correct

c) $\int \frac{dx}{\sin x(1+\cos x)}$

Let $t = \tan\frac{x}{2}$

$\therefore x = 2\tan^{-1} t$

$dx = \frac{2dt}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$

$= \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)}$

$= \int \frac{dt}{t\left(1+\frac{1-t^2}{1+t^2}\right)}$ ✓

$= \int \frac{1+t^2\ dt}{t(1+t^2+1-t^2)}$

$= \int \frac{1+t^2}{2t}dt$

$= \int \frac{1}{2t}+\frac{t}{2}\,dt$

$= \frac{1}{2}\ln|t| + \frac{t^2}{4} + C = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + \frac{\tan^2\left(\frac{x}{2}\right)}{4} + C$ ✓

d) $\int e^{-x}\cos x\,dx$

Let $u = \cos x$ $\frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = -\sin x$ $v = -e^{-x}$

$= (\cos x)(-e^{-x}) - \int(-e^{-x})(-\sin x)dx$

$= -e^{-x}\cos x - \int e^{-x}\sin x\,dx$ ✓

$u = \sin x$ $\frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = \cos x$ $v = -e^{-x}$

$I = -e^{-x}\cos x - \left[\sin x(-e^{-x}) - \int(-e^{-x})(\cos x)\frac{dx}{dx}\right]$ ✓

$I = -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x\,dx$

$2I = \left[e^{-x}(\sin x - \cos x)\right] + C_1$

$I = \frac{1}{2}e^{-x}(\sin x - \cos x) + C_2$ ✓

(e)(i) $\frac{d}{dx}\ln(x+\sqrt{x^2+4}) = \frac{1}{x+\sqrt{x^2+4}} \times \left(1+\frac{1}{2}(x^2+4)^{-\frac{1}{2}}\times 2x\right)$

$= \frac{1}{x+\sqrt{x^2+4}} \times \left(1+\frac{x}{\sqrt{x^2+4}}\right)$ ✓ SHOW

$= \frac{1}{(x+\sqrt{x^2+4})} \times \left(\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}}\right)$

$= \frac{1}{\sqrt{x^2+4}}$

(ii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx$

Substitute $u = \tan x$

$\therefore du = \sec^2 x\,dx$

x	$\frac{\pi}{4}$	$-\frac{\pi}{4}$
u	1	-1

$= \int_{-1}^{1} \frac{du}{\sqrt{u^2+4}}$

$= \left[\ln(u+\sqrt{u^2+4})\right]_{-1}^{1}$ ✓

$= \ln(1+\sqrt{5}) - \ln(-1+\sqrt{5})$

$= \ln\left(\frac{1+\sqrt{5}}{\sqrt{5}-1}\right)$

$= \ln\left[\frac{(1+\sqrt{5})(\sqrt{5}+1)}{5-1}\right] = \ln\frac{(1+\sqrt{5})^2}{4}$ ✓ SHOW

$= \ln\left(\frac{1+\sqrt{5}}{2}\right)^2$

$= 2\ln\left(\frac{1+\sqrt{5}}{2}\right)$

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13 a) $z\bar{z} = [Re(z+2)]^2$

Let $z = x+iy$. LHS $= z\bar{z} = (x+iy)(x-iy)$
$= x^2+y^2$

RHS $= [Re(x+2+iy)]^2$
$= (x+2)^2$ ✓

$\therefore x^2+y^2 = (x+2)^2$

$x^2+y^2 = x^2+4x+4$ ✓

$\therefore y^2 = 4(x+1)$

b)(i) $f(x) = \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) \quad (x>0)$

$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \times (-x^{-2})$

$= \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}} \; \frac{(\times x^2)}{(\times x^2)}$

$= \frac{1}{1+x^2} - \frac{1}{x^2+1}$

$= 0$ ✓

(ii) Since $f'(x) = 0 \quad f(x) = \text{constant}$

Sub $x = 1 \quad f(1) = \tan^{-1}1 + \tan^{-1}\left(\frac{1}{1}\right)$
$= \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2}$

hence $f(x) = \frac{\pi}{2}$

hence $\tan^{-1}2 + \tan^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$ ✓

c) Prove $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

STEP A

Show for $n=1$

LHS $= (\cos\theta - i\sin\theta)^1$

RHS $= \cos\theta - i\sin\theta$

$\therefore$ LHS = RHS Result is true for $n=1$.

STEP B Assume result is true for $n=k$, k positive integer

ie $(\cos\theta - i\sin\theta)^k = \cos k\theta - i\sin k\theta$

Attempt to show for $n=k+1$

ie $(\cos\theta - i\sin\theta)^{k+1} = \cos(k+1)\theta - i\sin(k+1)\theta$ ✓

LHS $= (\cos\theta - i\sin\theta)^k \times (\cos\theta - i\sin\theta)$ ✓

$= (\cos k\theta - i\sin k\theta) \times (\cos\theta - i\sin\theta)$ ✓ using assumption

$= \cos k\theta\cos\theta + i^2\sin k\theta\sin\theta - i\sin k\theta\cos\theta - i\cos k\theta\sin\theta$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) - i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k+1)\theta - i\sin(k+1)\theta$ ✓ using compound angle formulae.

$=$ RHS

$\therefore$ If result true for $n=k$, also true for $n=k+1$.

STEP C

By principle of mathematical induction, result proven for all positive integer n.

d) Given $-2\sin A\sin B = \cos(A+B) - \cos(A-B)$

$\int_0^{\frac{\pi}{4}} \sin 7x \sin 3x \, dx$

$= \int_0^{\frac{\pi}{4}} -\frac{1}{2}[\cos 10x - \cos 4x] \, dx$ ✓

$= -\frac{1}{2}\left[\frac{1}{10}\sin 10x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}}$

$= -\frac{1}{2}\left[\left(\frac{1}{10}\sin\frac{5\pi}{2} - \frac{1}{4}\sin\pi\right) - (0)\right]$

$= -\frac{1}{20}$ ✓

e) $\frac{x^2}{9}+\frac{y^2}{4}=1$ Diff implicitly w.r.t. x

$\frac{2x}{9}+\frac{2y}{4}\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-2x/9}{2y/4}$

$=\frac{-4x}{9y}$ ✓

Gradient @ $\left(\frac{3\sqrt{2}}{2},\sqrt{2}\right)$ $m=\frac{-4\left(\frac{3\sqrt{2}}{2}\right)}{9\sqrt{2}}$

$=\left(-\frac{2}{3}\right)$ ✓

f)(i) $U_n=\int_0^{\pi/4}\tan^{2n}\theta\,d\theta$

$=\int_0^{\pi/4}\tan^{2n-2}\theta(\sec^2\theta-1)\,d\theta$

$U_n=\int_0^{\pi/4}\tan^{2n-2}\theta\sec^2\theta\,d\theta-\int_0^{\pi/4}\tan^{2(n-1)}\theta\,d\theta$ ✓

$U_n=\left[\frac{\tan^{2n-1}\theta}{2n-1}\right]_0^{\pi/4}-U_{n-1}$

$=\left[\frac{\tan^{2n-1}\left(\frac{\pi}{4}\right)}{2n-1}-\frac{\tan^{2n-1}0}{2n-1}\right]-U_{n-1}$ ✓

$U_n=\frac{1}{2n-1}-U_{n-1}\quad(n\geq1)$

(ii) $\int_0^{\pi/4}\tan^6\theta\,d\theta=\frac{1}{2\times3-1}-\int_0^{\pi/4}\tan^4\theta\,d\theta$

↑ $n=3$

$=\frac{1}{5}-\left[\frac{1}{2\times2-1}-\int_0^{\pi/4}\tan^2\theta\,d\theta\right]$ ✓

$=\frac{1}{5}-\frac{1}{3}+\left[\frac{1}{2\times1-1}-\int_0^{\pi/4}d\theta\right]$

$=\frac{1}{5}-\frac{1}{3}+1-\left[\theta\right]_0^{\pi/4}$

$=\frac{1}{5}-\frac{1}{3}+1-\frac{\pi}{4}=\frac{13}{15}-\frac{\pi}{4}$ ✓

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14 a)(i) $\sin(A+B)=\sin A\cos B+\cos A\sin B$ ①

$\sin(A-B)=\sin A\cos B-\cos A\sin B$ ②

①+② $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ ⊛

Let $C=A+B$ ③

$D=A-B$ ④

③+④∴ $C+D=2A$, $A=\frac{1}{2}(C+D)$

③−④ $C-D=2B$, $B=\frac{1}{2}(C-D)$

√ SHOW

∴ substituting into ⊛

$\sin C+\sin D=2\sin\frac{1}{2}(C+D)\cos\frac{1}{2}(C-D)$

(ii) $\sin3x+\sin x=\cos x$

∴ Let $C=3x$ $D=x\Rightarrow\frac{1}{2}(C+D)=2x,\ \frac{1}{2}(C-D)=x$

∴ $2\sin2x\cos x=\cos x$ ✓

$2\sin2x\cos x-\cos x=0$

$\cos x(2\sin2x-1)=0$

$\cos x=0$ or $\sin2x=\frac{1}{2}$

$x=\frac{(2n+1)\pi}{2}$ where $n\in\mathbb{Z}$ ✓

or

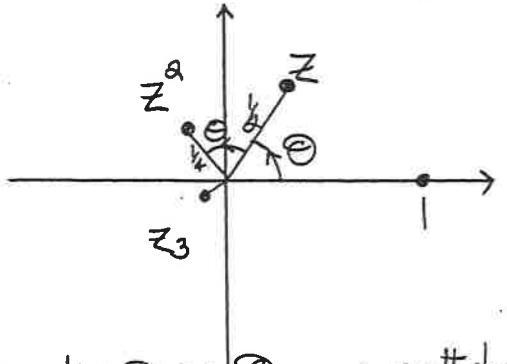
$2x=2n\pi+\frac{\pi}{6}$ or $(2n+1)\pi-\frac{\pi}{6}$

[Alternative: $2x=n\pi+(-1)^n\frac{\pi}{6}$]

$x=n\pi+\frac{\pi}{12}$ or $n\pi+\frac{5\pi}{12}$ ✓

[Alternative: $x=\frac{n\pi}{2}+(-1)^n\frac{\pi}{12}$]

b)(i)



✓

(ii) $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ ∴ multiplication by z halves the modulus and adds θ to the argument ✓

$z^n = \left(\frac{1}{2}\right)^n\left[\cos(\theta+\theta+\dots+\theta) + i\sin(\theta+\theta+\dots+\theta)\right]$

$= \left(\frac{1}{2}\right)^n(\cos n\theta + i\sin n\theta)$ ∴ $Im\{z^n\} = \left(\frac{1}{2}\right)^n \sin n\theta$

(iii) $1+z+z^2+z^3+\dots$ infinite GP $a=1$ $r=z$

$S_\infty = \frac{a}{1-r}$

$= \frac{1}{1-z}$

$= \frac{1}{1-\frac{1}{2}\text{cis}\theta}$ ✓

$= \frac{2}{2-\text{cis}\theta}$

$= \frac{2}{(2-\cos\theta)-i\sin\theta} \times \frac{(2-\cos\theta)+i\sin\theta}{(2-\cos\theta)+i\sin\theta}$ ✓

$= \frac{(4-2\cos\theta)+i2\sin\theta}{(4-4\cos\theta+\cos^2\theta)+\sin^2\theta}$

$= \frac{(4-2\cos\theta)+i2\sin\theta}{5-4\cos\theta}$ (*) ✓

∴ $Im\{S_\infty\} = \frac{2\sin\theta}{5-4\cos\theta}$

from (ii) $Im\{1+z+z^2+z^3+\dots\} = 0 + \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \dots$

hence $\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta = \frac{2\sin\theta}{5-4\cos\theta}$

c)(i) $\frac{x+2}{8(x^2+2x+2)} - \frac{x-2}{8(x^2-2x+2)}$

$= \frac{(x+2)(x^2-2x+2) - (x-2)(x^2+2x+2)}{8(x^2+2x+2)(x^2-2x+2)}$

$= \frac{(x^3-2x+4) - (x^3-2x-4)}{8(x^4-2x^3+2x^2+2x^3-4x^2+4x+2x^2-4x+4)}$

$= \frac{8}{8(x^4+4)}$ ✓ show

$= \frac{1}{(x^4+4)}$

(ii) $\int_0^N \frac{1}{(x^4+4)}dx = \int_0^N \frac{x+2}{8(x^2+2x+2)} - \frac{x-2}{8(x^2-2x+2)}\,dx$

$= \frac{1}{8}\int_0^N \frac{(x+1)+1}{(x+1)^2+1} - \frac{(x-1)-1}{(x-1)^2+1}\,dx$ ✓

$= \frac{1}{8}\int_0^N \frac{x+1}{(x+1)^2+1} + \frac{1}{(x+1)^2+1} - \frac{(x-1)}{(x-1)^2+1} + \frac{1}{(x-1)^2+1}\,dx$ ✓

$= \frac{1}{8}\left[\frac{1}{2}\ln\{(x+1)^2+1\} + \tan^{-1}(x+1) - \frac{1}{2}\ln\{(x-1)^2+1\} + \tan^{-1}(x-1)\right]_0^N$

$= \frac{1}{8}\left[\frac{1}{2}\ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right) + \tan^{-1}(x+1) + \tan^{-1}(x-1)\right]_0^N$

$= \frac{1}{16}\left[\ln\left(\frac{N^2+2N+2}{N^2-2N+2}\right) + 2\tan^{-1}(N+1) + 2\tan^{-1}(N-1) - \ln\left(\frac{2}{2}\right) + 2\tan^{-1}1 + 2\tan^{-1}(-1)\right]$

but $\tan^{-1}x$ an odd function ∴ $\tan^{-1}1 + \tan^{-1}(-1) = 0$ or $\frac{\pi}{4} + \left(-\frac{\pi}{4}\right) = 0$

$= \frac{1}{16}\left[\ln\left(\frac{N^2+2N+2}{N^2-2N+2}\right) - 2\tan^{-1}(N+1) + 2\tan^{-1}(N-1)\right]$ ✓

(iii) $f(x) = \frac{1}{x^4+4}$ is an even function, hence $\int_{-N}^{N} \frac{1}{x^4+4}\,dx = 2\int_0^N \frac{1}{x^4+4}\,dx$

$$\lim_{N\to\infty} \int_{-N}^{N} \frac{1}{x^4+4}\,dx = 2 \times \lim_{N\to\infty} \int_0^N \frac{1}{x^4+4}\,dx \quad ✓$$

$$= 2 \times \frac{1}{16}\left[\log 1 + 2\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{8} \times 2\pi$$

$$= \frac{\pi}{4} \quad ✓$$

since $\log\left(\frac{N^2+2N+2}{N^2-2N+2}\right) = \log\left(\frac{1+\frac{2}{N}+\frac{2}{N^2}}{1-\frac{2}{N}+\frac{2}{N^2}}\right)$ as $N \to \infty$ $\frac{2}{N}, \frac{2}{N^2} \to 0$

$\tan^{-1}(N+1)$ and $\tan^{-1}(N-1) \to \frac{\pi}{2}$

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